Combinatorial analytic tableaux in discrete optimization

Gennady Davydov
RAN, Institute of Transport Problems

Inna Davydova
St.Petersburg State University

P.O. Box 20, St.Petersburg-53, 199053, Russia
e-mail: inna@davy.usr.pu.ru

Abstract

Closed diagrams as generalization of some analytic tableaux are discussed. Application of closed diagram to search of optimum in discrete optimization problems is presented. Then new branch and bound modifications such as dual and nontreelike branch and bound methods, plait and bound method are advanced.

Keywords: dual systems, satisfiability, discrete optimization.

1 Introduction

Many structures such as acceptable mating of Andrews [1], connection graph of Bibel [2], combinatorial analytic tableaux of Cowen [3] in propositional calculus or dual systems of Lovasz [4] and Menger [5] in graph theory can be considered as closed diagram which was introduced in [6] (1971).

Diagram is a two colour hypergraph $\langle A, \mathbf{B}, \mathbf{D}\rangle$ where A is the set of vertices and both $\mathbf{B}$ and $\mathbf{D}$ is a family of hyperedges – subsets of A. Diagram is called *closed* if every minimal set π such that $\pi \cap b \neq \emptyset$ for every $b \in \mathbf{B}$ (in this case set π is called as *path in* $\mathbf{B}$) contains some set from family $\mathbf{D}$. It is convenient to represent sets from both $\mathbf{B}$ and $\mathbf{D}$ with columns and to depict a diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ as two-parts tableau $(\mathbf{B}|\mathbf{D})$.

Example 1. Let $A = \{a, b, c, d, e, f\}$. Then tableau

$$(B \mid D) = \left(\begin{array}{ccc|cccc} a & c & e & a & a & b & b \\ b & d & f & c & d & e & f \end{array}\right)$$

represents the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$. By fixing single element in each column from $\mathbf{B}$ we shall get a path in $\mathbf{B}$. The set $\{a, c, e\}$ provides an instance of a path. It easy to verify that each of all 8 paths contains some set from $\mathbf{D}$. Thus diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed.

Closed diagram may be interpreted in propositional calculus as follows. Let $p_i, i \in 1\colon m$, be propositional variables, A be the set of all literals and $S_j, j \in 1\colon n$, be subsets of A. Put $\mathbf{B} = \{\{p_i, \neg p_i\} \mid i \in 1\colon m\}$ and $\mathbf{D} = \{S_j \mid j \in 1\colon n\}$. Then *closedness* of diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is *equivalent* that *cnf*

$$F_1 = \bigwedge_{j \in 1:n} \bigvee_{a \in S_j} a$$

is *unsatisfiable* and *dnf*

$$F_2 = \bigvee_{j \in 1:n} \bigwedge_{a \in S_j} a$$

is *tautological.* In the formula F_1 (F_2) we consider every valuation as a set of literals with *false* (*true*) value. Then the statement "every path in $\mathbf{B}$ contains some set from $\mathbf{D}$" is the same as "every valuation is refutation for F_1" ("F_2 is valid for every valuation").

It follows in propositional case that closed diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ coincides with connection graph (matrix) of Bibel and acceptable mating of Andrews. Connections for matrix $\mathbf{D}$ are sets from $\mathbf{B}$.

Principal property of closed diagram is *duality*: if a diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed then the diagram $\langle A, \mathbf{D}, \mathbf{B}\rangle$ is closed also. Or in symmetric form: diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed iff every path in $\mathbf{B}$ has non-empty intersection with every path in $\mathbf{D}$. The proof is given in section 2.

Duality of closed diagrams makes possible to explain *resolution method* of Robinson [7] and *inverse method* of Maslov [8] in invariant terms [6]. See Section 2 in more details. Connection of closed diagrams with *dual systems* from graph theory [9]; *positive solvability* of systems $Cx = 0, x \geq 0$ for real-valued matrices C [10] and *probability* theory are briefly outlined at the same place.

The emphasis is on application of closed diagrams in discrete optimization problem (DOP) [11, 12, 13, 14, 15]. By DOP we mean the following problem.
Let $Q = B_1 \times B_2 \times \ldots \times B_n$, *where* B_i , $i \in 1{:}n$, *are given distinct finite sets, and* $f : Q \longrightarrow R^1$ *is a function defined on* Q. *It is required to search some* $q^* \in Q$ *such that* $f(q^*) = \max\{f(q) \mid q \in Q\}$.

Many well known discrete problems may be represented in the form of DOP. For example, knapsack problem (here $B_i = \{i, \neg i\}$ where i means that item i is putting in knapsack and $\neg i$ – no putting); plant location problem (here B_i is the set of possible variants of plant construction in the place i) and so on.

Every $q \in Q$ is called *solution*. If I is a fixed subset of the set of indices $\{1{:}n\}$ then every $\delta \in \prod_{i \in I} B_i$ is called as *partial solution* and designed as PS.

Let us interpret in diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ the family $\mathbf{B}$ as the family $\{B_i \mid i \in 1{:}n\}$ and the family $\mathbf{D}$ as some PS family. Then *closedness of the diagram is equivalent to completeness of* $\mathbf{D}$ *in the sense that every solution from* Q *is a extension of some PS from* $\mathbf{D}$ [13]. Section 3 contains representation of DOP by diagrams in more details.

We call PS family $\mathbf{D}$ *shutted* if set $R(\mathbf{D})$ of all extensions of PS from $\mathbf{D}$ is evaluated in such way that either it is coming to light that $q^* \notin R(\mathbf{D})$ or it is found q_0 such that $f(q_0) = \max\{f(q) \mid q \in R(\mathbf{D})\}$. If diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ for shutted $\mathbf{D}$ proves to be closed then optimal solution q^* is equal to q_0.

Primal and *dual* approaches can be used for search of $f(q^*)$: in primal (dual) complete (shutted) PS families $\mathbf{D}$ are constructed step by step and process will stop when current D becomes shutted (complete). Closed diagrams give possibility to realize both of these approaches. *Dual branch and bound method* as opposed to usual branch and bound method is described in Section 4. The *rule of closed diagrams generation* and on its basis *plait and bound method* as generalization of branch and bound method are presented in Section 5. Some conclusions about comparative efficiency of presented methods are contained in Section 6. Computer simulations have been made for knapsack problem.

2 Interpretations of closed diagrams

2.1 Duality

Define more precisely closed diagram.

Let A be a finite set and $\mathbf{B}$ and $\mathbf{D}$ be non-empty family of non-empty subsets of A. The tuple $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is called *diagram*. Every minimal set π such that $\pi \cap b \neq \emptyset$ for every $b \in \mathbf{B}$ is called *path* in $\mathbf{B}$. Diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is called *closed* if every path in $\mathbf{B}$ contains some set from family $\mathbf{D}$.

Theorem 1 (duality). *If diagram* $\langle A, \mathbf{B}, \mathbf{D}\rangle$ *is closed then diagram* $\langle A, \mathbf{D}, \mathbf{B}\rangle$ *is closed.*

Proof. Let exists a path ρ in $\mathbf{D}$ such that for every set $b \in \mathbf{B}$ would be $b \not\subseteq \rho$. Then we can define the path π in $\mathbf{B}$ such that $\pi \cap \rho = \emptyset$. Thus path π contains no one set from $\mathbf{D}$ that is in contradiction with closedness $\langle A, \mathbf{B}, \mathbf{D}\rangle$. $\square$

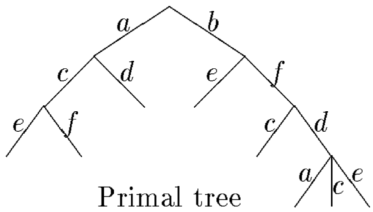
It follows from the proof that *diagram* $\langle A, \mathbf{B}, \mathbf{D}\rangle$ *is closed iff every path in* $\mathbf{B}$ *has non-empty intersection with every path in* $\mathbf{D}$.

To verify closedness of a diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ we may construct either *primal tree* or *dual tree*. In primal (dual) tree branching in every node is defined by some set from $\mathbf{B}$ ($\mathbf{D}$). For closed diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ every branch from primal (dual) tree contains a set from $\mathbf{D}$ ($\mathbf{B}$).

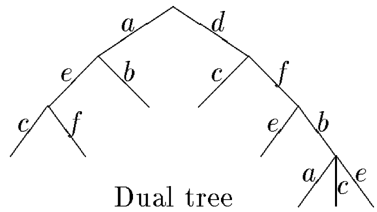
Example 2. Let $A = \{a, b, c, d, e, f\}$. Consider the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ presented by tableau

$$(B \mid D) = \left(\begin{array}{cccc|cccc} & & & a & & & & a \\ a & c & e & c & a & c & e & c \\ b & d & f & e & d & f & b & e \end{array}\right).$$

Then follows trees are instances of primal and dual trees.



Primal tree



Dual tree

Primal (dual) tree is a proof of closedness of the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ ($\langle A, \mathbf{D}, \mathbf{B}\rangle$).

2.2 Propositional calculus

Let $p_i, i \in 1\colon m$, be propositional variables, A be the set of all literals and $S_j, j \in 1\colon n$, be subsets of A. We shall consider the formula

$$F = \bigvee_{j \in 1:n} \bigwedge_{a \in S_j} a$$

Put $\mathbf{B} = \{\{p_i, \neg p_i\} \mid i \in 1\colon m\}$ and $\mathbf{D} = \{S_j \mid j \in 1\colon n\}$. Then *closedness* of diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is *equivalent* that F is *tautological.* Indeed, if we shall consider a valuation as a set of literals with *true* value then path in $\mathbf{B}$ would be defined. If every such path contains some set from $\mathbf{D}$ then would be disjunct in F with all *true* entries for every valuation and F would be tautology. Both primal and dual tree in this case gives a deduction for F and they are combinatorial analytic tableaux in the meaning [3].

The *R*esolution rule of Robinson [7] and rule B of Maslov's inverse method [8] can be considered as special cases of follows *ReB* rule [6, 16]. At first, some definitions.

Diagram $\langle A', \mathbf{B}', \mathbf{D}'\rangle$ is *subdiagram* of $\langle A, \mathbf{B}, \mathbf{D}\rangle$ if $A' \subseteq A, \mathbf{B}' \subseteq \mathbf{B}, \mathbf{D}' \subseteq \mathbf{D}$.

We shall say that a set s^r is *relaxation* of the set s if s^r is a subset of s.

A diagram $\langle A, \mathbf{B}^{\mathbf{r}}, \mathbf{D}\rangle$ ($\langle A, \mathbf{B}, \mathbf{D}^{\mathbf{r}}\rangle$) is called *left (right) relaxation* of the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ if the family $\mathbf{B}^{\mathbf{r}}$ ($\mathbf{D}^{\mathbf{r}}$) is getting from $\mathbf{B}$ ($\mathbf{D}$) by replacing some sets $b \in \mathbf{B}$ ($d \in \mathbf{D}$) with its relaxations b^r (d^r).

If left (right) relaxation $\langle A, \mathbf{B}^{\mathbf{r}}, \mathbf{D}\rangle$ ($\langle A, \mathbf{B}, \mathbf{D}^{\mathbf{r}}\rangle$) of the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed then the set $L = \bigcup_{b \in \mathbf{B}} b \setminus b^r$ ($R = \bigcup_{d \in \mathbf{D}} d \setminus d^r$) is called *left (right) bar.*

Now we can to define *ReB rule*:

Let $\langle A, \mathbf{B}, \mathbf{D}\rangle$ *be a diagram,* $\langle A', \mathbf{B}', \mathbf{D}'\rangle$ *be some its subdiagram. Suppose that closed left or right relaxation of this subdiagram exists. Let L or R be corresponding left or right bar.*

Add the bar L to family $\mathbf{B}$ (the bar R to family $\mathbf{D}$) and delete from the family $\mathbf{B}$ ($\mathbf{D}$) each set s such that the bar L (R) is a relaxation of s. Resulting diagram is called ReB-diagram for the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$.

If right relaxation is of the form $\left(\begin{smallmatrix} p_2 \\ \neg p_2 \end{smallmatrix}\Big| \begin{smallmatrix} p_2 & \neg p_2 \end{smallmatrix}\right)$ then ReB rule is being resolution rule and right bar R is a *clause.*

If left relaxation is of the form $(S_j^T|S_j)$ where $S_j^T = \{\{a\} \mid a \in S_j\}$ then ReB rule is B rule and left bar L is a F-favorable set [16].

Example 1.(continued) Let $F = (p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_3) \vee (\neg p_1 \wedge \neg p_3)$. Then $A = \{p_1, \neg p_1, p_2, \neg p_2, p_3, \neg p_3\}$ and corresponding to F diagram is presented by tableau

$$\left(\begin{array}{ccc|cccc} p_1 & p_2 & p_3 & p_1 & p_1 & \neg p_1 & \neg p_1 \\ \neg p_1 & \neg p_2 & \neg p_3 & p_2 & \neg p_2 & p_3 & \neg p_3 \end{array}\right)$$

which the same tableau as tableau in above example 1.

The diagram $\left(\begin{smallmatrix} p_2 \\ \neg p_2 \end{smallmatrix}\Big| \begin{smallmatrix} p_2 & \neg p_2 \end{smallmatrix}\right)$ is right relaxation of subdiagram $\left(\begin{smallmatrix} p_2 \\ \neg p_2 \end{smallmatrix}\Big| \begin{smallmatrix} p_1 & p_1 \\ p_2 & \neg p_2 \end{smallmatrix}\right)$, diagram

$$\left(\begin{array}{ccc|ccc} p_1 & p_2 & p_3 & \neg p_1 & \neg p_1 & \\ \neg p_1 & \neg p_2 & \neg p_3 & p_3 & \neg p_3 & p_1 \end{array}\right)$$

is ReB-diagram and right bar $R = \{p_1\}$ is a clause.

The diagram $\left(\begin{smallmatrix} \neg p_1 & p_3 \end{smallmatrix}\Big| \begin{smallmatrix} \neg p_1 \\ p_3 \end{smallmatrix}\right)$ is left relaxation of subdiagram $\left(\begin{smallmatrix} p_1 & p_3 \\ \neg p_1 & \neg p_3 \end{smallmatrix}\Big| \begin{smallmatrix} \neg p_1 \\ p_3 \end{smallmatrix}\right)$, diagram

$$\left(\begin{array}{cccc|cccc} p_1 & p_1 & p_2 & p_3 & p_1 & p_1 & \neg p_1 & \neg p_1 \\ \neg p_3 & \neg p_1 & \neg p_2 & \neg p_3 & p_2 & \neg p_2 & p_3 & \neg p_3 \end{array}\right)$$

is ReB-diagram and left bar $L = \{p_1, \neg p_3\}$ is F-favorable set.

Theorem 2. *A ReB-diagram for the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed $\Longleftrightarrow$ diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed.*

Proof. Suppose that ReB-diagram is got from $\langle A, \mathbf{B}, \mathbf{D}\rangle$ by right relaxation. Consideration in left case is the same.

At first, note that if we delete from $\mathbf{D}$ set such that its relaxation belongs to $\mathbf{D}$ also then closedness of diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ would be preserved. Indeed, if the path π in $\mathbf{B}$ contains a set that π contains its subset also.

It is obvious that if diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ be closed then by adding new set to $\mathbf{D}$ or $\mathbf{B}$ we shall get closed diagram again.

It remains to be seen whether ReB-diagram would being unclosed if diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is unclosed. By duality, in this case there is a path ρ in $\mathbf{D}$ containing no set from $\mathbf{B}$. The path ρ contains at least one element from the right bar R because relaxation of considering in ReB rule subdiagram is closed diagram. Hence ρ is a path in right part of ReB-diagram also and unclosedness is preserved. □

Diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed and formula F is a tautology iff by successive using ReB rule we shall get a ReB-diagram with empty left or right part finally.

2.3 Graph theory

Let $G = \langle V, E\rangle$ be a graph with two separated vertices v_1 and v_2. Then set $\mathbf{D}$ of all *joints* from v_1 to v_2 and set $\mathbf{B}$ of all *cuts* forms closed diagram $\langle E, \mathbf{B}, \mathbf{D}\rangle$. Well known flow problem may be stated in the following way. Let $\mathbf{D} = \{S_j, j \in N\}$ and for every $a \in E$ the capacity $c(a)$ is assigned. Then the function

$$f(c) = \max\left\{\sum\nolimits_{j \in N} y_j \;\Big|\; \sum\nolimits_{\{j | a \in S_j\}} y_j \le c(a), a \in E; y_j \ge 0, j \in N\right\}$$

is called *maximal flow*.

Well known Max-Flow-Min-Cut Theorem of L.Ford and D.Fulkerson is that

$$f(c) = \min\{\sum_{a\in\beta} c(a) \mid \beta \in \mathbf{B}\}$$

for all nonnegative assignments $c(a)$.

Let us replace set $\mathbf{D}$ of all joints and set $\mathbf{B}$ of all cuts by a families $\mathbf{D}$ and $\mathbf{B}$ in the diagram meaning and define $f(c)$ as above only with adding new conditions

$$\sum_{j\in N} y_j \leq \sum_{a\in\beta} c(a)\ ,\ \beta \in \mathbf{B}.$$

Then Max-Flow-Min-Cut Theorem is equivalent to closedness of the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ under the assumption that $|\beta \cap \delta| \leq 1$ for every $\beta \in \mathbf{B}$ and $\delta \in \mathbf{D}$. The proof is given in [9].

Special case of this generalization is flow problem defined on the Lovasz's pair (*clutter* $\mathbf{D}$, *blocker* $\mathbf{B}$) [4] where $\mathbf{D}$ is a family of sets such that every set is not a subset of another one and $\mathbf{B}$ is the family of all paths in $\mathbf{D}$.

Symmetric form of duality gives possibility to extend the notion "closed" on k-tuple $\langle A, \mathbf{B_1}, \mathbf{B_2}, \cdots, \mathbf{B_k}\rangle$ where as in the case of diagram each $\mathbf{B_i}$ is a family of subsets from A. Such k-tuple we call *k-gram* and k-gram $\langle A, \mathbf{B_1}, \mathbf{B_2}, \cdots, \mathbf{B_k}\rangle$ is called *closed* if $\bigcap_{i\in 1:k} \pi_i \neq \emptyset$ for all paths π_i in $\mathbf{B_i}$.

It has been found [9] that there is following association between graph *colouring* and *unclosedness* of k-gram. Let $G = \langle V, E\rangle$ be a graph and k be fixed positive integer. Put $\mathbf{B_i} = E, i \in 1\colon k$.

Theorem 3. *The k-gram* $\langle A, \mathbf{B_1}, \mathbf{B_2}, \cdots, \mathbf{B_k}\rangle$ *is unclosed* $\Longleftrightarrow$ *graph* G *is coloured by* k *colour.*

Proof. Every element from each $\mathbf{B_i} = E, i \in 1\colon k$ is being an edge $\{a, b\}$ jointing vertices a and b.

Let us suppose then graph G is coloured by k colours. Then there exists uncoloured by i colour vertex in every edge for every colour $i \in 1\colon k$. Form the set ρ_i of uncoloured by i colour vertices. The set ρ_i gives non-empty intersection with every edge $\{a, b\} \in \mathbf{B_i}$. Thus ρ_i contains a path π_i in $\mathbf{B_i}$. The intersection of all paths $\pi_i, i \in 1\colon k$ is empty because otherwise would be existing an uncoloured vertex. Thus k-gram $\langle A, \mathbf{B_1}, \mathbf{B_2}, \cdots, \mathbf{B_k}\rangle$ is unclosed.

On the other hand, let $\cap_{i\in 1:k}\pi_i = \emptyset$ for some paths $\pi_i, i \in 1\colon k$. Define complements $\pi_i' = V \setminus \pi_i, i \in 1\colon k$. It follows that $\cup_{i\in 1:k}\pi_i' = V$. Let us colour vertices $a \in \pi_i'$ by i colour (if $a \in \cap_{i\in I}\pi_i'$ for some $I \subset 1\colon k$ then vertex a is coloured by anyone colour from I). Then every vertex would be colouring and vertices of one edge would be colouring by distinct colours. □

Reduction of tautologies to 3-colouring via diagram was made in [17].

2.4 Positive solvability of linear systems

Real-valued matrix C is called *solvable* if the system

$$Cx = 0, \quad x \geq 0, \quad x \neq 0$$

has a solution and C is called *stable-solvable* if it is solvable and remains solvable for any variation of its elements preserving signs.

Solvable and stable solvable matrices plays an important part in the theory of linear economic models [18]. Solvable (stable-solvable) matrices describe (stable) equilibrium economic systems.

How solvable matrices are constructed?

Let C be a real-valued $m \times n$ matrix. Assign to its $j-$ column, $j \in 1{:}n$ the set S_j by including in this set literals i if $C_{ij} > 0$ and $\neg i$ if $C_{ij} < 0$. Put $A = \bigcup_{i \in 1:m} \{i, \neg i\}$; $\mathbf{B} = \{\{i, \neg i\} \mid i \in 1{:}m\}$; $\mathbf{D} = \{S_j \mid j \in 1{:}n\}$ and form the diagram $\langle A, \mathbf{B}, \mathbf{D} \rangle$.

Example 3. For every real-valued matrix C with scheme of signs

$$\begin{pmatrix} + & - & + & 0 & - \\ + & - & - & + & 0 \\ + & - & 0 & - & + \end{pmatrix}$$

the diagram $\langle A, \mathbf{B}, \mathbf{D} \rangle$ be of the form

$$\left(\begin{array}{ccc|cccccc} & & & 1 & \neg 1 & & & \\ 1 & 2 & 3 & 2 & \neg 2 & 1 & 2 & \neg 1 \\ \neg 1 & \neg 2 & \neg 3 & 3 & \neg 3 & \neg 2 & \neg 3 & 3 \end{array} \right)$$

Theorem 4. *A real-valued matrix C is stable-solvable $\iff$ its diagram $\langle A, \mathbf{B}, \mathbf{D} \rangle$ is closed.*

The proof is given in [10]. Based on separation theorem for convex sets (see for example [18]) new proof is given in [19].

In paper [19] it is proved that each *solvable matrix C with possible deleting and copying of some columns can be represented as a product LG of some matrix L and a stable-solvable matrix G.*

2.5 Probabilistic interpretation

Let $\langle A, \mathbf{B}, \mathbf{D} \rangle$ be a diagram such that $\beta_1 \cap \beta_2 = \emptyset$ for every $\beta_1, \beta_2 \in \mathbf{B}$, $\beta_1 \neq \beta_2$.

We shall consider every $a \in A$ as an elementary event with *positive* probability $p(a)$. Suppose that belonging to different sets $\beta \in \mathbf{B}$ events are independent. Interpret every set $\beta \in \mathbf{B}$ as the event consisting in that there is fulfilled one of inconsistent events $a \in \beta$. Thus probability $\mu(\beta)$ of this event is equal to $\sum_{a \in \beta} p(a)$. Suppose that

$$\mu(\beta) = \sum_{a \in \beta} p(a) = 1 \text{ for every } \beta \in \mathbf{B}.$$

In other words we interpret every set from family $\mathbf{B}$ as certain event.

Without loss of generality we shall suppose that

$$|\beta \cap \delta| \leq 1 \text{ for every } \beta \in \mathbf{B} \text{ and } \delta \in \mathbf{D}.$$

Then every set δ from family $\mathbf{D}$ is a subpath of some path in $\mathbf{B}$. And because of distinctions of sets in $\mathbf{B}$ we can represent δ as the event consisting in simultaneous fulfilling of all independent events $a \in \delta$. Thus the probability of this event is equal to

$$\mu(\delta) = \prod_{a \in \delta} p(a).$$

And finally we shall consider the family $\mathbf{D}$ as the event consisting in fulfilling at least one event δ from family $\mathbf{D}$. Let $\mu(\mathbf{D})$ be the probability of this event.

Theorem 5. *Diagram $\langle A, \mathbf{B}, \mathbf{D} \rangle$ is closed $\iff$ $\mu(\mathbf{D}) = 1$.*

Proof. Assign to every set $\delta \in \mathbf{D}$ the set $R(\delta)$ of all paths π in $\mathbf{B}$ such that $\delta \subseteq \pi$. Consider the set $R(\delta)$ as a event consisting in fulfilling at least one event π from family $R(\delta)$. All paths π from family $R(\delta)$ are inconsistent in pairs because events $a \in \beta$ are inconsistent for every $\beta \in \mathbf{B}$. Hence

$$\mu(R(\delta)) = \sum_{\pi \in R(\delta)} \mu(\pi) = \sum_{\pi \in R(\delta)} \prod_{a \in \pi} p(a) =$$

$$= \prod_{a \in \delta} p(a) \sum_{\pi \in R(\delta)} \prod_{a \in \pi \setminus \delta} p(a) = \prod_{a \in \delta} p(a) = \mu(\delta).$$

Last equality follows from that

$$\sum_{\pi \in R(\delta)} \prod_{a \in \pi \setminus \delta} p(a) = 1.$$

And this equality is proved by induction with number t of elements in the set $\pi \setminus \delta$. If $t = 1$ then we have $\sum_{a \in \beta} p(a) = 1$ for some $\beta \in \mathbf{B}$. In the case $\delta = \emptyset$ we have

$$\mu(\mathcal{P}) = \sum_{\pi \in \mathcal{P}} \prod_{a \in \pi} p(a) = 1$$

where $\mathcal{P}$ is the set of all paths in $\mathbf{B}$.

Thus probability $\mu(\delta)$ of fulfilling of the event δ is equal to the probability of fulfilling at least one of the inconsistent events from $R(\delta)$. It follows that probability of fulfilling at least one event δ from $\mathbf{D}$ is equal to the probability of fulfilling at least one of the events from $\bigcup_{\delta \in \mathbf{D}} R(\delta)$. That is

$$\mu(\mathbf{D}) = \mu(\bigcup_{\delta \in \mathbf{D}} R(\delta)) = \sum_{\pi \in \bigcup_{\delta \in \mathbf{D}} R(\delta)} \mu(\pi).$$

But from definition $R(\delta)$ it follows that $\bigcup_{\delta \in \mathbf{D}} R(\delta) \subseteq \mathcal{P}$ and

$$\mu(\bigcup_{\delta \in \mathbf{D}} R(\delta)) \leq \mu(\mathcal{P}) = 1.$$

It is not so hard to verify that closedeness of $\langle A, \mathbf{B}, \mathbf{D} \rangle$ is equivalent to that $\bigcup_{\delta \in \mathbf{D}} R(\delta) = \mathcal{P}$ which is to say that $\mu(\mathbf{D}) = 1$. □

It is a hard problem to define all paths from $\bigcup_{\delta \in \mathbf{D}} R(\delta)$. However to calculate $\mu(\bigcup_{\delta \in \mathbf{D}} R(\delta))$ we can use the formula of composite probability of dependent events [20]. Let for definiteness $\mathbf{D} = \{S_j \mid j \in 1\colon n\}$. Then this formula is of the form

$$\mu(\bigcup_{j \in 1:n} R(S_j)) = \sum_{t \in 1:n} (-1)^{t-1} \sum_{I \in P_n^t} \mu(\bigcap_{j \in I} R(S_j))$$

where P_n^t is the set of all combinations of n things t at a time.

Note that if there are in a combination I indices i and j such that there exist a set $\beta \in \mathbf{B}$ and distinct events $a_i, a_j \in \beta$ such that $a_i \in S_i$ and $a_j \in S_j$ then $R(S_i) \bigcap R(S_j) = \emptyset$ and $\mu(\bigcap_{j \in I} R(S_j)) = 0$ for such I. It follows from definition of the path in $\mathbf{B}$ and disjointness of sets from $\mathbf{B}$. In this case sets (or events) S_i and S_j are *inconsistent.* Otherwise sets S_i and S_j are *consistent.* Moreover a combination I is called *consistent* if sets S_i and S_j are consistent for all pair $\{i, j\}$ from I.

Only consistent combinations I produce nonzero items in the last sum. To select those items define *the graph of consistency* $G = \langle N, E \rangle$: the set of its vertices is the set of indices $N = 1\colon n$, and two vertices i, j are linked with an edge if sets S_i and S_j are consistent.

Then nonzero items will be produced only by complete subgraphs of the graph of consistency. A complete subgraph of the graph of consistency is characterized by the set $I \subseteq N$ of its vertices and by its *size* $t(I) = |I|$.

It follows from the definition of $R(S_j)$ that

$$\bigcap_{j\in I} R(S_j) = \bigcap_{j\in I}\{\pi \in \mathcal{P} \mid S_j \subseteq \pi\} = \{\pi \in \mathcal{P} \mid \bigcup_{j\in I} S_j \subseteq \pi\} = R(\bigcup_{j\in I} S_j)$$

for complete subgraph with vertices set I. Put $S_I = \bigcup_{j\in I} S_j$. Then from the proof of theorem 5 we have

$$\mu(\bigcap_{j\in I} R(S_j)) = \mu(R(S_I)) = \prod_{a\in S_I} p(a).$$

Design $\mathcal{G}_t$ the set of all complete subgraphs of the graph of consistency with size t. Then finally the formula for calculation of $\mu(\mathbf{D})$ is

$$\mu(\mathbf{D}) = \mu(\bigcup_{j\in 1:n} R(S_j)) = \sum_{t\in 1:n}(-1)^{t-1}\sum_{I\in\mathcal{G}_t}\prod_{a\in S_I} p(a).$$

Example 3.(continued) Consider the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ from this example. Put

$$p(i) = p(\neg i) = 1/2 \text{ for every } i = 1, 2, 3.$$

All sets in $\mathbf{D}$ are inconsistent in pair. Thus there are only 5 separated vertices in the graph of consistency. Each complete subgraph consists only single vertex. Then,

$$\mu(\mathbf{D}) = (1/2)^3 + (1/2)^3 + (1/2)^2 + (1/2)^2 + (1/2)^2 = 1$$

what is meant by theorem 5 that diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is closed.

Theorem 5 gives *criterion of closedness* of the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$. By using last formula we can select a polynomial class of verifiable by this criterion diagrams. It consists in such diagrams that number of complete subgraphs of its graph of consistency polynomial grows with growth of number of sets in $\mathbf{D}$. For example diagrams with its *graph of consistency* being *bipartite* or *planar*.

3 Representation of discrete optimization problem by closed diagrams

3.1 Completeness and shuttedness of partial solutions family

Let discrete optimization problem (DOP), solutions and partial solution (PS) are defined as in Introduction:

By DOP we mean following problem.

Let $Q = B_1 \times B_2 \times \ldots \times B_n$, where B_i , $i \in 1{:}n$, are given distinct finite sets, and $f : Q \longrightarrow R^1$ is a function defined on Q. It is required to search some $q^* \in Q$ such that $f(q^*) = \max\{f(q) \mid q \in Q\}$.

Every $q \in Q$ is called *solution* and q^* is called *optimal solution.* If I is fixed subset of the set of indices $\{1{:}n\}$ then any $\delta \in \prod_{i\in I} B_i$ is called as *partial solution* and designed as PS.

Since all sets $B_i, i \in 1{:}n$, are distinct then we can consider every solution–vector $q = q[1{:}n] \in B_1 \times B_2 \times \ldots \times B_n$ as the set $q = \{q[i] \mid i \in 1{:}n\}$ and every PS–vector $\delta = \delta[I] \in \prod_{i\in I} B_i$ as the set $\delta = \{\delta[i] \mid i \in I\}$.

With every PS δ we shall connect the set $R(\delta) = \{q \in Q \mid \delta \subseteq q\}$ of all extensions of δ.

Let D be a PS family and $R(D) = \cup_{\delta \in D} R(\delta)$. The family D is called *complete* if $R(D) = Q$.

Suppose that every PS δ is evaluated by the estimation $E(\delta)$ so that it holds

$$E(\delta) \geq f(q) \;,\; q \in R(\delta).$$

Let $r = f(q_0)$ where q_0 is some solution. PS δ is called *shutted* if

$$E(\delta) \leq r$$

and PS family D is called *shutted* if every $\delta \in D$ is shutted and $q_0 \in R(D)$.

If PS family D is shutted then the solution q_0 gives local optimum in the area $R(D)$. If we were fortunate to construct PS family D which turns out both shutted and complete then DOP would be solved and q_0 would be optimal solution.

Now for search of optimal solution we can use two approaches: *primal* and *dual*. In primal (respectively dual) approach complete (respectively shutted) PS families D are constructed step by step and process come to a stop when current D becomes shutted (respectively complete).

Well known branch and bound method realizes primal approach in which each family D is the family of all branches in PS tree. Closed diagrams make it possible to construct "nontreelike" PS families in primal approach and to realize dual approach.

3.2 Closed diagrams in DOP

Let $\mathbf{B}$ in diagram $\langle A, \mathbf{B}, \mathbf{D} \rangle$ be interpret as DOP family $\{B_1, B_2, \ldots, B_n\}$ and $\mathbf{D}$ as some PS family. From definition of closed diagram it follows that if diagram is closed then $\mathbf{D}$ is complete PS family.

So we can realize primal approach by crossing from one closed diagram $\langle A, \mathbf{B}, \mathbf{D} \rangle$ to another. In every such diagram the family $\mathbf{B}$ is the same. But the family $\mathbf{D}$ is changed so that set of all solutions Q would be covering by sets of solutions decreasing step by step. It means that PS in $\mathbf{D}$ will be extended step by step. Process will stop when all PS from $\mathbf{D}$ would be shutted. Usual branch and bound method uses a primal tree as family $\mathbf{D}$. Use could be made of "nontreelike" families $\mathbf{D}$. In Section 5 we shall consider technique for nontree-binding of PS families.

But at first in next Section we consider how closed diagrams are used in realization of dual approach.

4 Dual branch and bound method

Let PS family D is shutted. It follows from duality that to verify completenes of D we should verify all paths in D. If each path contains a set from DOP family $\mathbf{B} = \{B_i \mid i \in 1\colon n\}$ then D is complete. Otherwise we shall add to D new sets for doing D "more complete". Show how this can be doing in the case when each $B_i = \{i, \neg i\}$.

Suppose that we can construct shutted PS family D by some heuristic considerations in DOP. Then *dual branch and bound method* [13] is started with this D and consists in follows steps.

1. Let π be a path in D such that $B_i \in \pi$ for no $i \in 1\colon n$.

   If such path no exists then stop. Solution q_0 defining shuttedness of D will be optimal solution in this case.

   Else go to 2.

2. Replace every entry in π by its negation. Obtained set $I(\pi)$ we call *inversion.* In the family $D \cup I(\pi)$ each across inversion $I(\pi)$ continuation of the path π contains a set B_i.

3. Consider inversion $I(\pi)$ as PS and evaluate it.

   If PS $I(\pi)$ is shutted then we have new shutted PS family $D \cup I(\pi)$. Put $D := D \cup I(\pi)$ and go to 1.

   Else PS $I(\pi)$ is continued to be shutted. Let PS δ be shutted continuation of PS $I(\pi)$. In the worst case PS δ will be a solution. Put $D := D \cup \delta$ and go to 1.

5 Plaits and bound method

5.1 Nontreelike closed diagrams

To verify closedness of a diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ we can construct either primal or dual tree (see 2.1 Duality). With every primal tree T we connect the *value* $v(T)$ – the number of branches in T. Let $\mathcal{T}$ be the set of all primal trees for the diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$. Put $v^* = \min\{v(T) \mid T \in \mathcal{T}\}$. Primal tree T with $v(T) = v^*$ is called *minimal tree* and v^* is called *minimal value.* Closed diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is called *treelike* (*nontreelike*) if $v^* = |\mathbf{D}|$ (respectively $v^* > |\mathbf{D}|$).

In the example 1 family $\mathbf{D}$ itself consists of all branches from minimal tree. So minimal value $v^* = 4 = |\mathbf{D}|$ and the diagram is treelike.

Depicted primal tree from the example 2 is minimal tree and minimal value $v^* = 8 > 4 = |\mathbf{D}|$. Thus in this case the diagram is nontreelike.

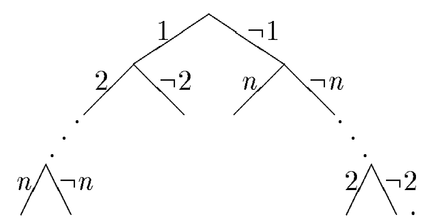
PS family $\mathbf{D}$ is called *nontreelike* if $\langle A, \mathbf{B}, \mathbf{D}\rangle$ with $\mathbf{B} = \{B_i \mid i \in 1{:}n\}$ be nontrelike closed diagram.

Next example may be useful for construction nontreelike PS families in DOP where each $B_i = \{i, \neg i\}$.

Example 4. Let $A = \bigcup_{i \in 1:n}\{i, \neg i\}$. Consider diagram $\langle A, \mathbf{B}, \mathbf{D}\rangle$ presented by tableau

$$\left(\begin{array}{cccc|cccccc}
 & & & & 1 & \neg 1 & & & & \\
 & & & & 2 & \neg 2 & & & & \\
1 & 2 & & n & \vdots & \vdots & 1 & 2 & & n-1 & n \\
\neg 1 & \neg 2 & \cdots & \neg n & n & \neg n & \neg 2 & \neg 3 & \cdots & \neg n & \neg 1
\end{array}\right).$$

It is not so hard to verify that minimal tree $\mathbf{T}$ in this case is of the form



Thus we have $v^* = v(\mathbf{T}) = 2n > (n+2) = |\mathbf{D}|$ for $n > 2$ and $\langle A, \mathbf{B}, \mathbf{D}\rangle$ is nontreelike closed diagram.

By using nontrelike complete family $\mathbf{D}$ we can construct PS tree such that a complete family $\mathbf{D}$ defines branching in every node of this tree. We call this generalization of usual PS tree as *Kowalski tree.* Some branching in Kowalski tree may be repeated in contrast with usual tree. But in algorithmic realization it is possible to avoid such repetitions. Every subtree of Kowalski tree would be supply a nontrelike PS family if nontreelike families are used for branching. Follows technique allows to bind nontreelike families by another way.

5.2 Plaits and D–product

The set $\Omega = \{D_1, \ldots, D_m\}$ of families we shall call *plait.* Every $\omega \in D_1 \times \ldots \times D_m$ is a sequence $\omega = \{\delta_1, \ldots, \delta_m\}$ of sets. Let $\delta(\omega) = \cup_{i=1,\ldots,m}\, \delta_i$.

Family $P(\Omega) = \{\delta(\omega) \mid \omega \in D_1 \times \ldots \times D_m\}$ is called *product* of plait Ω. Product is the special case of *D-product* which is defined as follows.

1. Let $\Omega = \{D_1, \ldots, D_m\}$ be a plait. Correspond to every $\delta \in D_i$ its own name $a(\delta)$ from some set A.

2. Let $X_i = \{a(\delta) \mid \delta \in D_i\}$ be the set of D_i names and $\langle A, \mathbf{X}, \mathbf{Y}\rangle$ where $\mathbf{X} = \{X_i \mid i \in 1\colon m\}$ be some closed diagram. It is called *pattern.*

3. Correspond the set $\delta(y) = \cup_{a(\delta)\in y}\delta$ to every names set $y \in \mathbf{Y}$. The family $D = \{\delta(y) \mid y \in \mathbf{Y}\}$ is called *D-product* of plait Ω.

Example 5. Let $\Omega = \{D_1, D_2, D_3\}$ be the plait with
$D_1 = \{\{a, b\}, \{c, d\}\}, \quad D_2 = \{\{e\}, \{a\}\}, \quad D_3 = \{\{e\}, \{b\}\}$. I we put
$X_1 = \{1, \neg 1\}, \quad X_2 = \{2, \neg 2\}, \quad X_3 = \{3, \neg 3\}$
and use closed diagram from example 4 for n=3 as pattern $\langle A, \mathbf{X}, \mathbf{Y}\rangle$ by setting
$\mathbf{X} :=\mathbf{B}$, $\mathbf{Y} :=\mathbf{D}$ then D-product of plait Ω is the family
$D = \{\{a, b, e\}, \{c, d, a, b\}, \{a, b\}, \{e, b\}, \{e, c, d\}\}$.

Theorem 6. *If every D_i in the plait $\Omega = \{D_1, D_2, \ldots, D_m\}$ is complete PS family then D-product of this plait is complete PS family.*

The proof consists in direct application of definition of closed diagram [15].

By using of D-product technique we can consider plait as generalized branch. To work with a plait as with usual branch we must to evaluate its. Recall that PS family D is shutted if every $\delta \in D$ is shutted. Plait $\Omega = \{D_1, D_2, \ldots, D_m\}$ will be shutted if its product $P(\Omega)$ considering as PS family is shutted. If we would be in position to evaluate plait (all PS from $P(\Omega)$) broadly without consideration its product but via common analysis of families D_i then using of plaits could be efficient.

Moreover we can consider the generalization of D-product which consists in partition of PS families by subfamilies – blocks and in naming of PS blocks but not single PS. Then a plait of blocks will be corresponded to every names set $y \in \mathbf{Y}$. With such plait we can to work separately. By replacing in branch and bound method branch with plait of blocks and PS tree by this block generalization of D-product we shall obtain *plait and bound method* [15].

Next example illustrates how D-product may be used in forming new nontreelike closed diagram from given nontreelike closed diagram.

Example 6. Let set Q in DOP is defined as

$$Q = \prod_{i\in 1:n} (\{a_i, x_i\} \times \{b_i, y_i\} \times \{b_i, y_i\}).$$

Put

$$\mathbf{B}_i = \begin{pmatrix} a_i & b_i & c_i \\ x_i & y_i & z_i \end{pmatrix}; \qquad \mathbf{D}_i = \begin{pmatrix} a_i & x_i & & & \\ b_i & y_i & a_i & b_i & c_i \\ c_i & z_i & y_i & z_i & x_i \end{pmatrix}.$$

Then $(\mathbf{B}_i \mid \mathbf{D}_i)$ presents a special case of nontreelike diagram from Example 4.

Split $\mathbf{D}_i$ at two parts

$$v_i = \begin{pmatrix} a_i & x_i \\ b_i & y_i \\ c_i & z_i \end{pmatrix} \quad \text{and} \quad w_i = \begin{pmatrix} a_i & b_i & c_i \\ y_i & z_i & x_i \end{pmatrix}$$

and consider pattern

$$(\mathbf{X} \mid \mathbf{Y}) = \left(\begin{array}{cccc|cccccc} v_1 & v_2 & & v_n & v_1 & w_1 & w_2 & & w_{n-1} & w_n \\ w_1 & w_2 & \cdots & w_n & & v_2 & v_3 & \cdots & v_n & \end{array} \right).$$

The application of D-product replaces every entry in $\mathbf{Y}$ by its columns. After this every column in $\mathbf{Y}$ would be a plait. Replace plait $\binom{w_i}{v_{i+1}}$, $i \in \{1{:}\, n-1\}$, by usual product $\mathbf{G}_i = w_i \times v_{i+1}$. Then diagram

$$(\mathbf{B}^n \mid \mathbf{D}^n) = (\mathbf{B}_1\mathbf{B}_2 \ldots \mathbf{B}_n \mid v_1\mathbf{G}_1 \ldots \mathbf{G}_{n-1}w_n)$$

would be closed.

Let q_n be the value of minimal tree for this diagram. In [15] it was proved that $q_1 = 6$, $q_2 = 16$ and for $n > 2$

$$q_n = q_1 + (q_2 - q_1)\Big(\sum_{i=2}^{n-1} F_{i-1} \cdot 2^{i-2} + (n - F_k + 1) \cdot 2^{k-2}\Big)$$

where F_{i-1} and F_k are **Fibonacci** numbers ($F_0 = F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$) and $F_k \le n < F_{k+1}$.

From evaluation $F_k = O(\alpha^k)$ where $\alpha \approx 1.6$ with regard to the last inequality we have an evaluation $q_n = O(n^{2.5})$. However $|\mathbf{D}^n| = 2 + 6(n-1) + 3 = 6n - 1$. Thus $q_n > |\mathbf{D}^n|$ and diagram $(\mathbf{B}^n \mid \mathbf{D}^n)$ would be nontreelike.

6 Conclusion

Usual branch and bound, branch and bound with Kowalski tree and dual branch and bound methods are compared for knapsack problem. Testing was executed by A.Kosenko. More then 1000 tests were performed with knapsack problems of size 100.

For the most part *dual branch and bound method* has gone up both by time and by the number of evaluated PS. This is because very good estimation and ordering of items are known in knapsack problem. And dual method quickly proves optimality of heuristic found solution by search of refutations. Primal method proves optimality by search of best solutions.

We hope that nontreelike primal technique will be good for problems in which it is hard to guess right optimal solution and plait technique will be able for problems in which partial solutions are joined in blocks with common characteristic by semantic way. May be well known quadratic assignment problem is a such problem.

References

[1] P.B. Andrews *Refutations by matings* — IEEE Transactions on Computer **25** (1976), 801–807

[2] W. Bibel *Automated Theorem Proving* — Braunschweig (1982)

[3] R.H. Cowen *Combinatoirial analytic tableaux* — Reports on mathematical logic, **27** (1993), 29–39

[4] L. Lovasz *An algorithmic Theory of Numbers, Graphs and Convexity* — SIAM Philadelphia (1986)

[5] T.A. McKee *Dual properties within graph theory* — Fundamenta mathematicae (1987), v.128/2, 91–97

[6] G.V. Davydov *The synthesis of the resolution method and the inverse method* (Russian) — Zapiski Nauchnykh Seminarov Lomi, **20** (1971), 24–35 (English translation in Journal of Sov. Math. **1**, no.1 (1973), 12–18)

[7] J.A. Robinson *On automatic deduction* — Rice Univ. Studies **50** (1)(1964), 69–89

[8] S.Ju. Maslov *The inverse method of establishing deducibility in the classical predicate calculus (Russian)* — Dokl. Akad. Nauk. SSSR **159** (1964), 17–20 (English translation in Sov. Math. Dokl. **5** (1964), 1420–1424)

[9] G.V. Davydov, I.M. Davydova *A flow interpretation of NP-complete problems* (Russian) — Izvestya VUZov, Math. (1988), no.12, 59–61 (English translation in Sov. Math. (Iz. Vuz.), **32** (1988), no.12, 96–101)

[10] G.V. Davydov, I.M. Davydova *Tautologies and positive solvability of linear homogeneous systems* — Annals of Pure and Applied Logic **57** (1992), 27–43

[11] I.M. Davydova *Enumeration schemes in location problems* (Russian) — Textbook (1985), Leningrad, 100p.

[12] G.V. Davydov, I.M. Davydova *Dual criterion of optimality in discrete problems* (Russian) — Izvestya VUZov, Math. (1985), no.1, 57–59

[13] G.V. Davydov, I.M. Davydova *Dual algorithms in discrete optimization* (Russian) — Voprosy cybernetiki, **131** (1987), 90–117

[14] G.V. Davydov, I.M. Davydova *Nontreelike coverings in discrete optimization* (Russian) — Izvestya VUZov, Math. (1988), no.3, 66–69 (English translation in Sov. Math. (Iz. Vuz.), **32** (1988), no.3, 93–97)

[15] G.V. Davydov, I.M. Davydova *Plait and bound method* (Russian) — Issled. oper. i stat. model., **6** (1994), 14–30

[16] G. Mints *Proof Theory in the USSR 1925 - 1969* — Journal of Symbolic Logic **56** (2)(1991), 385 - 424

[17] G.V. Davydov, P.Yu. Suvorov *Reduction of tautologies to 3 colouring graph* (Russian) — Zapiski nauchn. semin. LOMI **40** (1974), 10–13

[18] D.Gale *The theory of linear economic models* — McGraw-Hill Book Company, New York, 1960.

[19] G.Davydov, I.Davydova *The solvable matrices* — (Russian) Vestnik SPbU, Mathematics (1993), no.1, 3–9.

[20] W. Feller *An introduction to probability theory and its applications* — New York(1950)